Answers to Sample Questions

No calculator was used.

1. (a) Let the particular person be p_1. Let the other people be p_2, p_3, p_4, p_5, p_6 and p_7. So the sample space is $S = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$. Then the probability that p_1 is selected is $Pr(\{p_1\}) = |\{p_1\}|/|S| = 1/7$.

   (b) Let the females be f_1, f_2, f_3 and f_4, and let the males be m_1, m_2 and m_3. Then the sample space is $S = \{f_1, f_2, f_3, f_4, m_1, m_2, m_3\}$. So $Pr(male) = |\{m_1, m_2, m_3\}|/|S| = 3/7$.

   (c) As in (a), let the particular person be p_1 and the others be p_2, p_3, p_4, p_5, p_6 and p_7. Then the sample space is the Cartesian product of $\{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ with itself; i.e., $S = \{(p_1, p_1), (p_1, p_2), \ldots, (p_1, p_7), (p_2, p_1), (p_2, p_2), \ldots, (p_2, p_7), \ldots, (p_7, p_1), (p_7, p_2), \ldots, (p_7, p_7)\}$. So $Pr(p_1 \text{ twice}) = Pr\{(p_1, p_1)\} = |\{(p_1, p_1)\}|/|S| = 1/49$.

   Alternatively, the event of p_1 being chosen on the first day is (clearly) independent of the event that he or she is chosen on the second day. So the probability of p_1 being chosen on both days is $(1/7) \cdot (1/7) = 1/49$.

   (d) As in (b), let the females be f_1, f_2, f_3 and f_4, and let the males be m_1, m_2 and m_3. Then the sample space is the Cartesian product of $\{f_1, f_2, f_3, f_4, m_1, m_2, m_3\}$ with itself, which has elements such as (f_1, f_1), (f_2, m_5), and so on. There are 49 such ordered pairs. (For example, the ordered pair (f_2, m_5) occurs if female f_2 is the person selected on the first day and male m_5 is the person selected on the second day.) Now the Cartesian product of $\{f_1, f_2, f_3, f_4\}$ with itself is a 16-element subset of the sample space S, and the Cartesian product of $\{m_1, m_2, m_3\}$ with itself is a 9-element subset of S. The ordered pairs in these two subsets represent all the possible ways in which two males or two females are selected on successive days. So the probability of the two people having the same gender is $(16 + 9)/49 = 25/49$.

2. (a) $Pr(\text{score} > 4) = Pr(\{5, 6\}) = 2/6 = 1/3$.

   (b) $Pr(\text{score} < 2) = Pr(\{1\}) = 1/6$. Since the two outcomes are disjoint, the probability of one or the other happening is the sum of their individual probabilities. That is, $Pr(\text{score} > 4 \text{ or score} < 2) = 1/3 + 1/6 = 1/2$.

3. The sample space is the Cartesian product of $\{1, 2, 3, 4, 5, 6\}$ with itself, which is a set of 36 ordered pairs.

   (a) $(1/6)(1/6) = 1/36$

   (b) The probability of each of 1 twice, 2 twice, ..., 5 twice is the same as the probability of 6 twice, which is 1/36. So the probability that the same score occurs twice is $6 \cdot (1/36) = 1/6$.

   (c) $1 - 1/6 = 5/6$

   (d) One way is to list the ordered pairs that represent this outcome. They are $(1, 4)$, $(2, 5)$, $(3, 6)$, $(4, 1)$, $(5, 2)$ and $(6, 3)$. So the probability of this outcome is $6/36 = 1/6$.

   (e) The relevant ordered pairs are $(2, 6)$, $(3, 5)$, $(4, 4)$, $(5, 3)$ and $(6, 2)$. Since there are five of them, the probability of this outcome is $5/36$.

   (f) Careful thought (or listing the relevant ordered pairs) shows that this happens half the time. So the probability is $1/2$.

   (g) Let A mean "the scores are the same", and B mean that "they are both 5". Clearly B is a one-element subset of the 6-element set A. So $A \cap B = B$. Therefore $Pr(B|A) = Pr(A \cap B)/Pr(A) = Pr(B)/Pr(A) = (1/36)/(6/36) = 1/6$.

(h) Let A mean that the scores are different, and B mean that they differ by 2 or more. From (c), the probability of the two scores differing is $5/6 = 30/36$ so that there are 30 ways in which the scores can differ. That is, $|A| = 30$. Now B consists of $(1, 3)$, $(1, 4)$, $(1, 5)$, $(1, 6)$, $(2, 4)$, $(2, 5)$, $(2, 6)$, $(3, 5)$, $(3, 6)$, $(4, 6)$ and the reverse ordered pair of each of these. So $|B| = 20$. Since B is a subset of A, the probability of B given A equals $|B|/|A|$ which equals $2/3$.

(i) Let A mean "both odd" and B mean "differ by 2 or more". Then $Pr(B|A) = Pr(A \cap B)/Pr(A)$. Now the probability that they are both odd is $Pr(A) = (1/2) \cdot (1/2) = 1/4$, since the two events of "odd first" and "odd second" are independent. Also, $A \cap B = \{(1, 3), (1, 5), (3, 1), (3, 5), (5, 1), (5, 3)\}$ so that $Pr(A \cap B) = 6/36 = 1/6$. Therefore $Pr(B|A) = (1/6)/(1/4) = 2/3$.

(j) Let A and B be as in the answer to (i) above. Now $\sim B$ is the event that the scores don't differ by two or more, which means that they differ by less than two. In other words, either they are the same or they differ by 1. So $\sim B = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4), (4, 5), (5, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}$. Therefore $|\sim B| = 16$, so that $|B| = 36 - 16 = 20$. So $Pr(B) = 20/36 = 5/9$. Then $Pr(A|B) = (1/6)/(5/9) = 3/10$.

(k) No, since $Pr(A \cap B) = 1/6$ but the product of $Pr(A)$ and $Pr(B)$ is $(1/4) \cdot (5/9) = 5/36$.

4. (a) The sample space has 12 ordered pairs, 6 having T as the second coordinate and the other 6 having F as the second coordinate. The ordered pair $(4, H)$ is one of the 12 equiprobable outcomes, whence its probability is $1/12$.

(b) Let A mean "4 on the die" and B mean "H on the coin". Then $A \cap B = \{(4, H)\}$ which occurs with probability $1/12$. Also $Pr(A) = 1/6$ and $Pr(B) = 1/2$. So $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B) = 1/6 + 1/2 - 1/12 = 7/12$.

(c) We have seen that $Pr(A) = 1/6$, $Pr(B) = 1/2$ and $Pr(A \cap B) = 1/12$. So $Pr(A \cap B) = Pr(A) \cdot Pr(B)$. Thus the two outcomes are independent.