Answers to Sample Questions

No calculator was used.

1. $|X \cup Y| = |X| + |Y| = 8$ (since X and Y are disjoint)

2. (a) Since the possible outcomes from tossing the coin are H (for head) and T (for tail), the two events (tossing a coin and selecting a compass direction) are disjoint. So the Sum Rule can be used. Let C be the coin and B be the box. Then the number of outcomes is

$$n(C) + n(B) = 2 + 4 = 6.$$

Alternatively we can use the notation of Set Theory. If C is the set $\{\text{H,T}\}$ of coin outcomes and B is the set $\{\text{N,S,E,W}\}$ of box outcomes, then the total number of possible outcomes is

$$|C \cup B| = |C| + |B| = 2 + 4 = 6$$

since $C \cap B = \varnothing$.

(b) Let D be the die. Since the outcomes from the three events are pairwise disjoint, the Sum Rule applies. The number of outcomes is

$$n(C) + n(D) + n(B) = 2 + 6 + 4 = 12.$$

Alternatively, let D be the set $\{1, 2, 3, 4, 5, 6\}$ of die outcomes. Then the total number of outcomes is

$$|C \cup D \cup B| = |C| + |D| + |B|$$

since $C \cap D = C \cap B = D \cap B = \varnothing$.

3. (a) Applying the Product Rule gives

$$n(C) \cdot n(B) = 2 \cdot 4 = 8.$$

Alternatively, the set of outcomes is the Cartesian product

$$C \times B = \{(H, N), (H, S), (H, E), (H, W), (T, N), (T, S), (T, E), (T, W)\}$$

so that the number of outcomes is

$$|C \times B| = |C| \cdot |B| = 2 \cdot 4 = 8.$$

(b) Applying the Product Rule gives

$$n(C) \cdot n(D) \cdot n(B) = 2 \cdot 6 \cdot 4 = 48.$$

Alternatively:

$$|C \times D \times B| = |C| \cdot |D| \cdot |B| = 2 \cdot 6 \cdot 4 = 48.$$

4. (a) Since the sets of outcomes are no longer disjoint, the Sum Rule can no longer be used. Sometimes an alternative rule called the Sum Rule with Overlaps is used. In Set-Theoretic terms, here is the answer:

$$|C \cup B| = |C| + |B| - |C \cap B| = 2 + 5 - 1 = 6$$

(b) We use the formula for the cardinality of the union of three sets:

$$\begin{aligned}|C \cup D \cup B| &= |C| + |D| + |B| - |C \cap D| - |C \cap B| - |D \cap B| + |C \cap D \cap B| \\ &= (2+6+5) - (0+1+2) + 0 = 10\end{aligned}$$

(c) We use the formula for the cardinality of the Cartesian product of three sets:

$$|C \times D \times B| = |C| \cdot |D| \cdot |B| = 2 \cdot 6 \cdot 5 = 60.$$

5. (a) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

   (b) ${}^6\mathrm{P}_4 = \dfrac{6!}{2!} = 360$

   (c) ${}^6\mathrm{C}_4 = {}^6\mathrm{C}_2 = \dfrac{6 \cdot 5}{2 \cdot 1} = 15$

6. (a) $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

   (b) ${}^8\mathrm{C}_3 = \dfrac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$

   (c) ${}^8\mathrm{P}_3 = \dfrac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336$

   (d) ${}^{16}\mathrm{C}_{14} = {}^{16}\mathrm{C}_2 = \dfrac{16 \cdot 15}{2 \cdot 1} = 120$

7. (a) Since there are 366 different possible birthdays, if 366 or fewer people came then it's possible that there would be no duplicate birthdays. But if 367 or more people came then the Pigeonhole Principle ensures that there would be two people with the same birthday. (Here the pigeons are the guests and the pigeonholes are the birthdays.)

   (b) The answer is 23. (This will be explained later.)

   (c) With 6 or fewer selections, it's possible that there are no pairs. But with 7 or more selections, the Pigeonhole Principle ensures that there would be two ear muffs of the same colour. (Here the pigeons are the ear muffs and the pigeonholes are the colours.)

   (d) With 5 or fewer selections, it's possible that there are no same-square pairs. (For example, she might select 0, 1, 2, 3 and 4.) But with 6 or more selections, the Pigeonhole Principle ensures that there would be two integers with the same square. (Here the pigeons are the integers and the pigeonholes are the sets $\{-4,4\}$, $\{-3,3\}$, $\{-2,2\}$, $\{-1,1\}$ and $\{0\}$.)