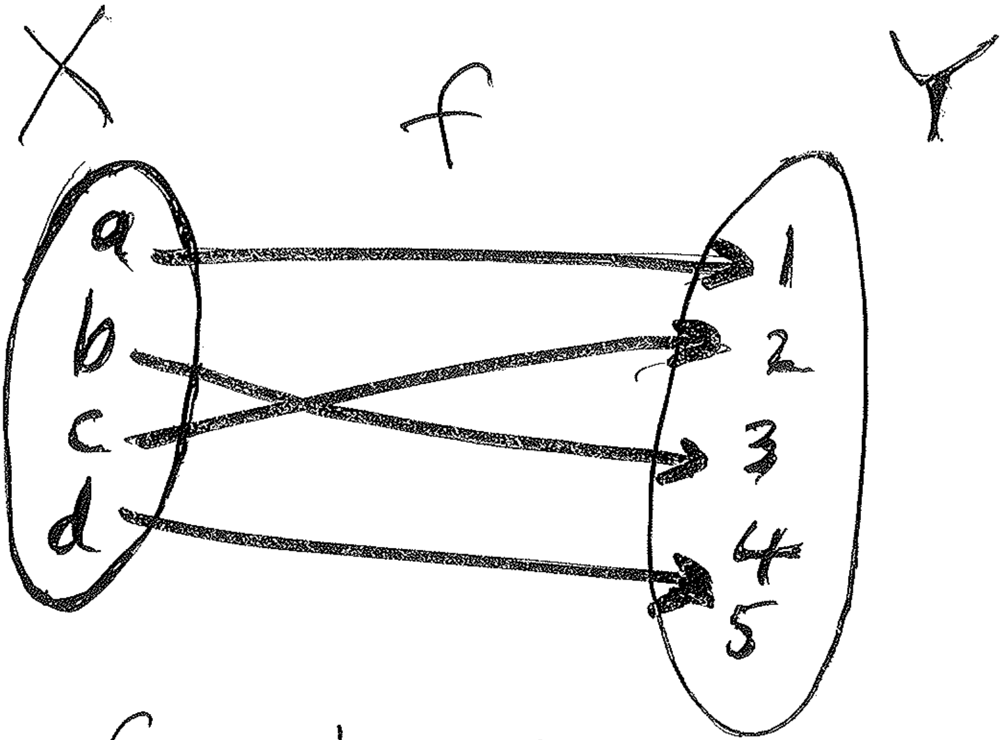
Lecture 6

In this lecture we consider two important properties exhibited by some functions: the "one-to-one" property and the "onto" property.

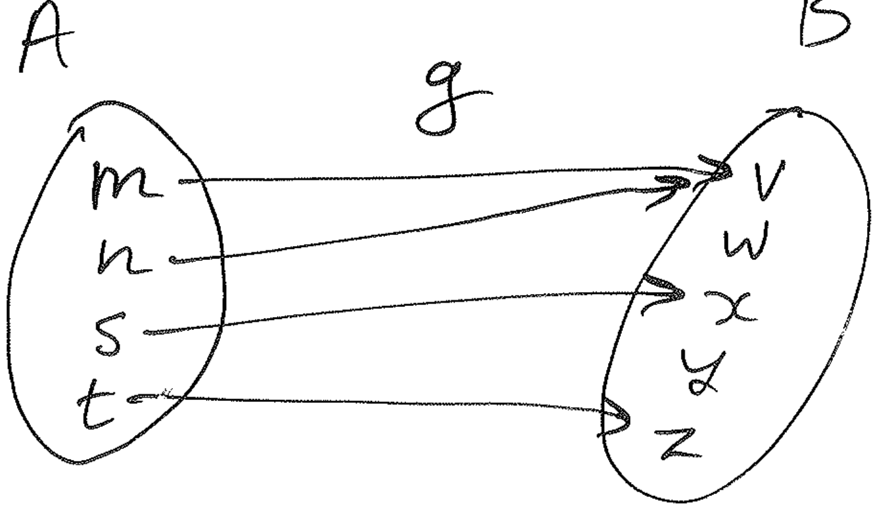


This function is an example of a one-to-one function.

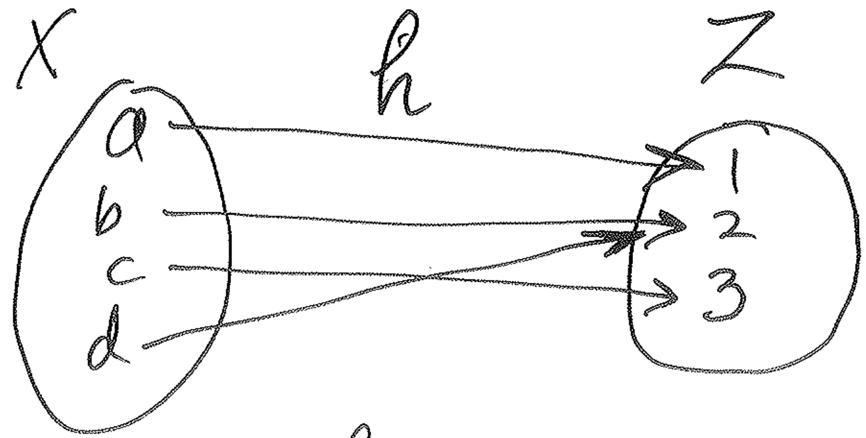
A function $f: X \longrightarrow Y$ is <u>one-to-one</u> (or <u>injective</u>, or an <u>injection</u>) if distinct elements of X have distinct images in Y.

That is, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

E.g. Here, g is not one-to-one.



We see that $g(m) = g(n) = v$. That is, m and n have the same image.



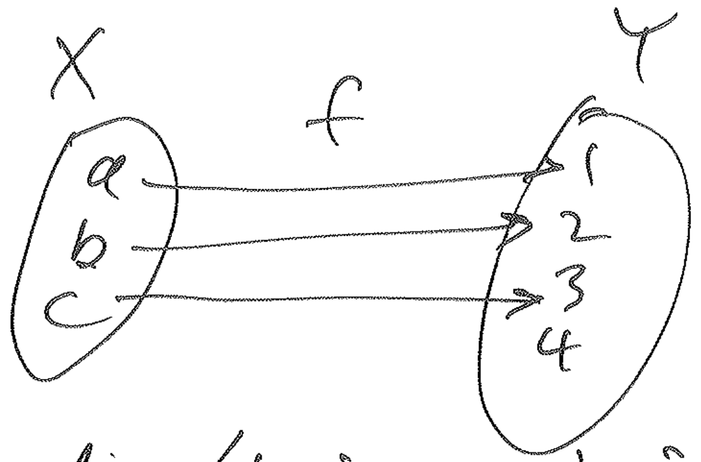
The function h is an example of an "onto" function. Every element of Z receives at least one arrow.

A function $f: X \to Y$ is <u>onto</u> (or <u>surjective</u>, or a <u>surjection</u>) if $f(X) = Y$;

i.e., the range is the whole of the codomain; i.e., for every element $y \in Y$ there exists $x \in X$ such that $f(x) = y$.

In the above example, h is not one-to-one.

Eg. <u>Not</u> onto.



Note that 4 is not in the range.

If $f(x)=y$, so that y is the image of x (under the action of f), we also say that x is a <u>pre-image</u> of y.

So a function is onto if every element of the codomain has a pre-image.

A function which is both injective (one-to-one) and surjective (onto) is said to be <u>bijective</u> or <u>a bijection</u>.

For finite sets X and Y, there exists an injection from X to Y if and only if $|X| \leq |Y|$.

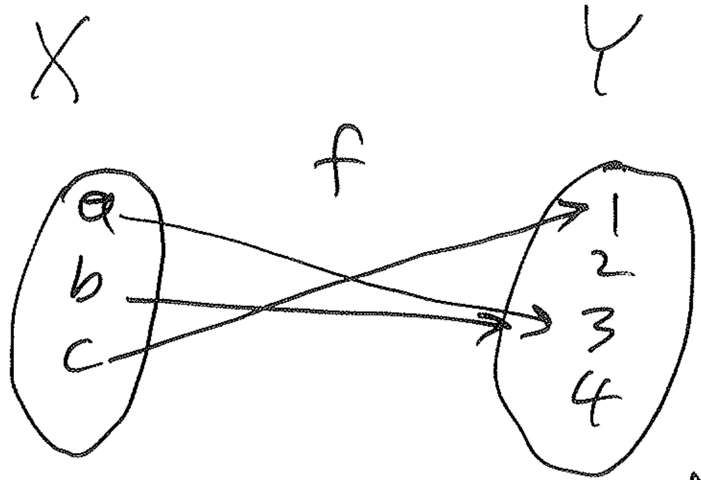
There exists a surjection from X onto Y if and only if $|X| \geq |Y|$.

$\therefore$ there exists a bijection if and only if $|X| = |Y|$.

A bijection is also called a "one-to-one correspondence".

Examples

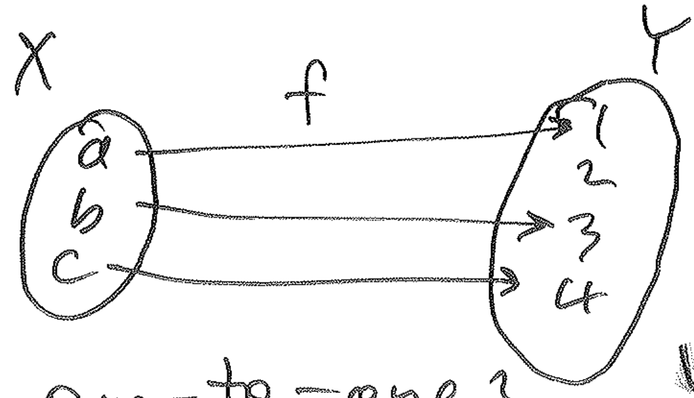
①



one-to-one? NO

onto? NO

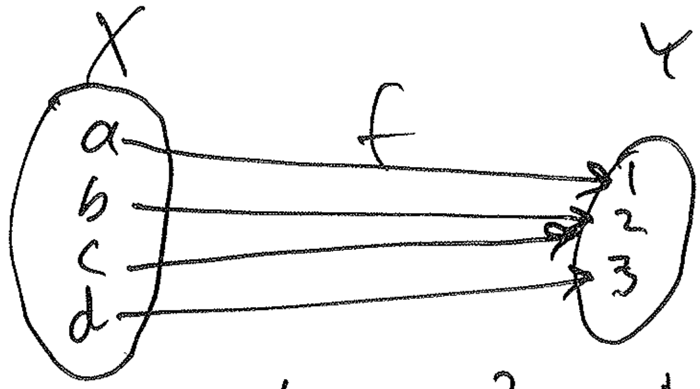
②



one-to-one? Yes

onto? No

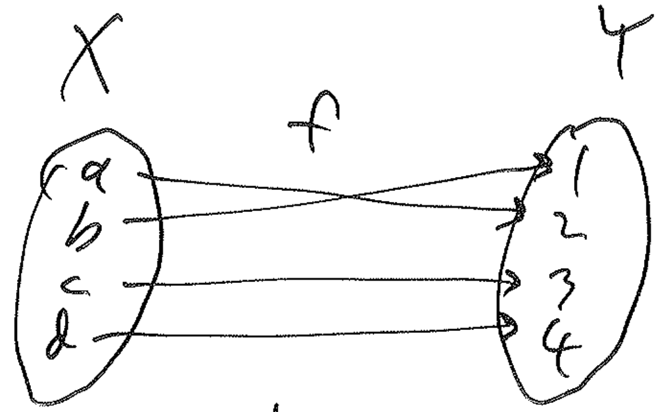
(3)



one-to-one? NO

onto? Yes

(4)



one-to-one? Yes

onto? Yes

Infinite Sets

Two infinite sets are regarded as having the same cardinality (size) if there exists a bijection between them.

Eg. There are exactly as many integers as there are positive integers.

$$\begin{array}{ccc}
\underline{\mathbb{N} \setminus \{0\}} & f & \underline{\mathbb{Z}} \\
1 & \longrightarrow & -1 \\
2 & \longrightarrow & 1 \\
3 & \longrightarrow & -2 \\
4 & \longrightarrow & 2 \\
5 & \longrightarrow & -3 \\
6 & \longrightarrow & 3 \\
\vdots & & \vdots
\end{array}$$

Clearly f is a bijection.

Also,

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|.$$

The size of those sets is called <u>countable infinity</u>.

In contrast, $\mathbb{R}$ is a much bigger infinite set whose size is <u>uncountably infinite</u>.