Lecture 4

We continue our study of relations on sets, and introduce the idea of an equivalence relation.

Recall that a relation R on a set X is any subset of X^2 (i.e., any ~~set~~ of ordered pairs (x,y) where $x,y \in X$).

Properties of Relations

Let ρ be a relation on X.

lower case "rho"

We say ρ is reflexive if

$$(x,x) \in \rho \quad \text{for all } x \in X.$$

We say ρ is symmetric if

$$(x,y) \in \rho \implies (y,x) \in \rho.$$

We say ρ is transitive if

$$\left.\begin{array}{l}(x,y) \in \rho \\ (y,z) \in \rho\end{array}\right\} \Rightarrow (x,z) \in \rho.$$

Instead of

$$(x,y) \in \rho$$

we often write

$$x \rho y$$

and say "x rho y" or "x is ρ-related to y".

Reflexive: $x \rho x$ for all x

<u>Symmetric</u>: $x \rho y \Rightarrow y \rho x$

Transitive: $\left.\begin{matrix} x \rho y \\ y \rho z \end{matrix}\right\} \Rightarrow x \rho z$

<u>E.g.</u> Let ρ on $\mathbb{R}$ mean

$$x + y > 1.$$

Is this refl., symm., or trans.?

Sol^{n}

Ref? No.

Eg., $0+0 \not> 1$

$\therefore (0,0) \notin \rho$

Symm? Yes.

Suppose $(x,y) \in \rho$.

Then $x+y > 1$.

$\therefore y+x > 1$

$\therefore (y,x) \in \rho$.

Trans?

Suppose $x+y > 1$

and $y+z > 1$.

Can we conclude that $x+z > 1$?

No. Eg., $x=0, y=5, z=-1$.

$x+y = 5 > 1$

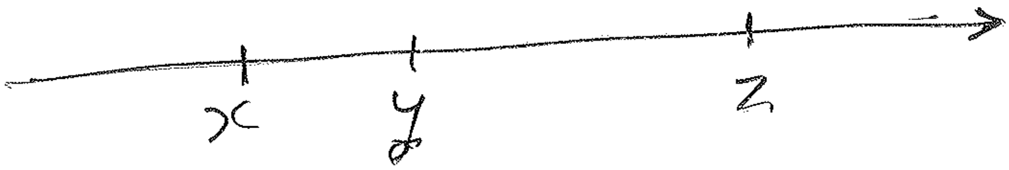
$y+z = 4 > 1$

But $x+z = -1 < 1$.

Can we think of a well known transitive relation?

$<$

Note that if $x<y$ and $y<z$ then $x<z$.



Equivalence Relations

A relation ρ on a set X is said to be an equivalence (relation) when it is reflexive, symmetric and transitive.

some positive integer

We say

$$a \equiv b \pmod{n}$$

(a is congruent to b modulo n)

when $(a-b)$ is an integer multiple of n.

Usually we want a and b to be integers.

$$a \equiv b \pmod{n}$$

means

$$a - b = tn$$ for some integer t.

<u>Eg.</u> $X = \{1, 2, \ldots, 10\}$

Define ρ on X by

$x \rho y$ if $x \equiv y \pmod{3}$.

Write down ρ as a set of ordered pairs.

Sol$^{\underline{n}}$

$$\rho = \{(1,1),(1,4),(1,7),(1,10),$$
$$(2,2),(2,5),(2,8),$$
$$(3,3),(3,6),(3,9),$$
$$(4,1),(4,4),(4,7),(4,10),$$
$$\vdots$$
$$(9,3),(9,6),(9,9),$$
$$(10,1),(10,4),(10,7),(10,10)\}$$

It can be shown that the relation

$$a \equiv b \pmod{n}$$

is always an equivalence relation on $\mathbb{Z}$ and its subsets.

$$a \equiv a \pmod{n}$$

$$a \equiv b \pmod{n} \Rightarrow b \equiv a \pmod{n}$$

$$\left.\begin{array}{l} a \equiv b \pmod{n} \\ b \equiv c \pmod{n} \end{array}\right\} \Rightarrow a \equiv c \pmod{n}$$

---

4 o'clock

$+$ 12 hours

$=$ "1600 hours"

$=$ 16 o'clock (24-hour time)

$=$ 4 o'clock (12-hour time)

this is because

$$4 \equiv 16 \pmod{12}.$$

Coding theory is based on arithmetic modulo 2.

Partitions

A <u>partition</u> is a set of disjoint, nonempty subsets of a given set X whose union is X.

Essentially, a partition divides X into subsets.

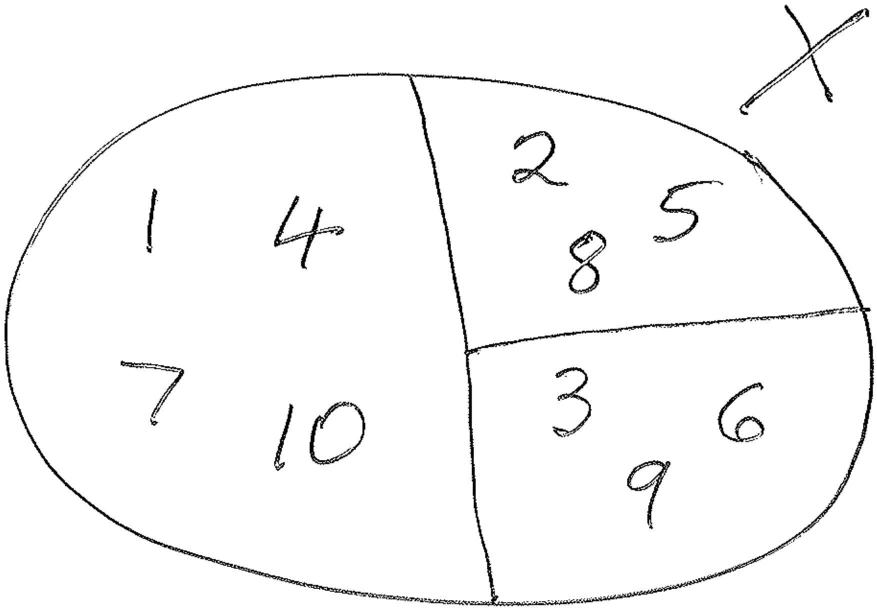
<u>Theorem</u>

For every equivalence relation there is a corresponding partition, and vice versa.

<u>Proof</u>

Not examinable.

<u>E.g.</u> Last example.



The partition corresponding to p is often denoted by π_p.

Here:

$$\pi_p = \{ \{1,4,7,10\}, \{2,5,8\}, \{3,6,9\} \}.$$